

DYNAMICS OF THE SUBSETS OF NATURAL NUMBERS: A NURSERY RHYME OF CHAOS

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Abstract

We introduce a metric on the set of all subsets of natural numbers which converts it into a cantor set. On this set endowed with the metric, we introduce a very simple map which exhibits chaotic behaviour. This simple map is almost like a nursery rhyme of chaos.

The set of all subsets of natural numbers is a much larger set than the set of natural numbers. In fact, it has the cardinality of the set of real numbers. We shall denote the set of all subsets of natural numbers by the symbol Σ , set of natural numbers by \mathbf{N} , subsets of natural numbers by α, β etc.

$$\begin{aligned}\alpha &= (n_1, n_2, \dots, n_k, \dots) \\ \beta &= (m_1, m_2, \dots, m_k, \dots)\end{aligned}\tag{1}$$

Here n_k and m_k are natural numbers and $\alpha, \beta \subset \mathbf{N}$ and $\in \Sigma$. On the set Σ , we define symmetric set theoretic difference as

$$\begin{aligned}\gamma &\equiv (\alpha - \beta) \\ &= (\alpha \cup \beta) \setminus (\alpha \cap \beta) \\ &= (\alpha \setminus \beta) \cup (\beta \setminus \alpha)\end{aligned}\tag{2}$$

We now define on the set Σ a norm as follows (1) To the null set \emptyset , we associate zero i.e., $\|\emptyset\| = 0$ (2) To the set $\alpha = (n_1, n_2, \dots, n_k)$, we associate either

$$\|\alpha\| = \sum_{n_k \in \alpha} (1/M^{n_k})\tag{3}$$

where $M \geq 2$. In fact, we take $M = 2$. or

$$\|S_1\| = \sum_{n_k \in \alpha} (1/n_k!)\tag{4}$$

In this paper, we will use the first definition of the norm though we might as well have used the second one. The distance between two subsets α and β is given by

$$\begin{aligned}\|(\alpha - \beta)\| &\equiv \|\gamma\| \\ &= \|(\alpha \cup \beta) \setminus (\alpha \cap \beta)\| \\ &= \|(\alpha \setminus \beta) \cup (\beta \setminus \alpha)\|\end{aligned}\tag{5}$$

It is clear from the definitions that $\|\gamma\| = 0$ imply $\alpha = \beta$, and the triangle inequality is easily satisfied. On this set Σ endowed with above mentioned metric, we introduce the "Reduction by Unity" map Ω , as follows

$$(n_1, n_2, \dots, n_k, \dots) \rightarrow (n_1 - 1, n_2 - 1, \dots, n_k - 1, \dots) \quad (6)$$

if none of $n_k = 1$. If one of n_k , say $n_1 = 1$, then

$$(1, n_2, \dots, n_k, \dots) \rightarrow (n_2 - 1, \dots, n_k - 1, \dots) \quad (7)$$

It is clear from the definition that the "Reduction by Unity" map Ω is two-to-one.

The above structures defined on the set Σ are very natural. We explain this by the following physical example. Let us consider an infinite number of boxes. Any number of boxes can be empty or filled with particles. However, the filled boxes are such that no two boxes have the same number of particles. The configuration of the particles in the boxes can be represented by the sequences $(n_1, n_2, \dots, n_k, \dots)$ where $n_1, n_2, \dots, n_k, \dots$ are distinct positive integers. Note that we do not have zero as an entry in the sequences even though there are any number of empty boxes because for the process to be described in the following these empty boxes are irrelevant except when all of them are empty. We describe a process of evaporation of particles from the boxes in the following way: at a time from each box a single particle is evaporated. In other words, we define evaporation per unit time via the map $(n_1, n_2, \dots, n_k, \dots) \rightarrow (n_1 - 1, n_2 - 1, \dots, n_k - 1, \dots)$ (if $n_1 = 1$, then there is no entry at the location of $n_1 - 1$). Note that this map is two-to-one. We arrive at the configuration $(n_1 - 1, n_2 - 1, \dots, n_k - 1, \dots)$ either from the evaporation of the configuration $(n_1, n_2, \dots, n_k, \dots)$ or $(1, n_1, n_2, \dots, n_k, \dots)$. The dynamics of the evaporation can not be analysed any more unless we introduce some kind of distance or metric on the set of all possible configuration of the particles in the boxes. This will tell us how far or close particle configurations are from each other. An adhoc mathematical definition would do the job. However, we shall try to be as close as possible to a physical situation. It is natural to assume that two configurations with similar number of particles in the boxes are closer to each other than those with different number of particles. The more the dissimilar number of entries in the two configurations the further they are. Another assumption we would like to make is that two configurations differing from each other in very large entries are not too far from each other. This sounds a bit

unnatural. However, it is required mathematically. Note that the configuration \emptyset , corresponding to "no particle" in any of the boxes also belongs to the set of all possible configurations. Therefore, the distance of any configuration from the null configuration \emptyset essentially defines a norm. Now if the distance between two configurations keep increasing with larger entries of particles in the boxes there will be problem with convergence of the norm. Therefore, the two requirement mentioned above are necessary for defining a proper metric. This metric introduces an interesting scenario for the evaporation process: there is a temporal scale separation- the short time and long time behaviour of the evaporation is different in the sense that two generic configurations which stay close to each other in short time separate exponentially from each other in the long time. In fact, what will be proven in this paper is that the set of possible configurations of particles in the boxes (i.e., the set of all subsets of natural numbers Σ) is a Cantor set and the evaporation process (i.e., the "Reduction by Unity" map Ω on Σ) is chaotic.

A set is called a Cantor set if it is closed, totally disconnected and is a perfect set. "Closed" means every sequence of its' points converges to some point within the set. "Totally disconnected" means it contains no interval. A perfect set means every point in it is an accumulation point. The classical example of a Cantor set is the "Cantor middle-thirds" set. The "Cantor middle-thirds" set is nicely described by Cantor map

$$\begin{aligned} F(x) &= 3x \quad \text{for} \quad -\infty < x \leq 1/2 \\ &= 3(1-x) \quad \text{for} \quad 1/2 < x \leq \infty \end{aligned} \tag{8}$$

on the real line \mathbf{R} . Note that most points of the real line \mathbf{R} go to $-\infty$ under the iteration of the Cantor map. However, there exists a set which is a subset of the interval $\mathbf{I}=[0,1]$ that does stay within the interval \mathbf{I} and is invariant under the iteration of the map $F(x)$. This is precisely the "Cantor middle-thirds" set. The Cantor map $F(x)$ on this set is chaotic [1,2]. This means that,

- (1) The map has the property of sensitive dependence on initial conditions.
- (2) It has dense set of periodic orbits.

(3) It has a topologically transitive (chaotic) trajectory i.e., a trajectory which goes arbitrary close to every point in the set.

We shall proof that the set of all subsets of natural numbers, Σ endowed with the metric introduced in begining of paper, is **homeomorphic** to the Cantor set. In other words there exists, between the "Cantor middle-thirds" set and the set of all subsets of natural numbers, Σ , a mapping which is one-one, onto and continuous along with the the inverse. We shall also proof that the Cantor map on Cantor set is topologically conjugate to the "Reduction by Unity" map on Σ .

From the definition of the Cantor map, note that $F'(x) = 3$, and this means that the map is expansive. Moreover, $F^{-1}(I)$, where $I = [0, 1]$, consists of two subintervals $I_0 = [0, 1/3]$ and $I_1 = [2/3, 1]$. Also for any subinterval $J \subset I$, $F^{-1}(J)$ consists of two subintervals, one in I_0 and the other in I_1 . The length of the subinterval J is 3-times the length of either of the subintervals $F^{-1}(J)$. Also, $F^{-n}(I_0)$ and $F^{-n}(I_1) \rightarrow 0$ as 3^{-n} . These properties of the Cantor map can be used to establish a homeomorphism between Λ , the Cantor set and the set Σ .

Let us construct the itinerary $S(x)$ of a point x under the Cantor map $F(x)$,

$$S(x) = (n_1, n_2, \dots, n_k, \dots) \quad (9)$$

The symbols n_j are introduced in the following way: $n_j \in S(x)$ if $F^{n_j-1}(x) \in I_0$. It will be argued that $S(x)$ defined above is a homeomorphism. Firstly, it is obvious that to every point $x \in \Lambda$, where Λ is the Cantor set, there is sequenc $S(x)$ i.e., a subset of Σ .

Now we proof that to every subset of Σ there is a point $x \in \Lambda$. Let $\eta = (1, 2, \dots, n)$ and $\xi, \zeta \subset \eta$. $\xi \cup \zeta = \eta$ and $\xi \cap \zeta = \emptyset$, the nullset. Define ξ_1 and ζ_1 as two sets obtained from ξ and ζ by "Unit Reduction map". Therefore, $\xi_1 \cup \zeta_1 = \eta_1$, where $\eta_1 = (1, 2, \dots, n-1)$ and $\xi_1 \cap \zeta_1 = \emptyset$, the nullset. Define,

$$I_{\xi, \zeta}^n = \{x \in I \mid F^{k-1}(x) \in I_0 \text{ for } k \in \xi, F^{k-1}(x) \in I_1 \text{ for } k \in \zeta\}$$

$$= \left(\bigcap_{k \in \xi} F^{-k+1}(I_0) \right) \left(\bigcap_{k \in \zeta} F^{-k+1}(I_1) \right) \quad (10)$$

Let us proof that $I_{\xi, \zeta}^n$ is a connected interval. First note that integer, $1 \in \xi$ or $1 \in \zeta$, therefore,

$$I_{\xi, \zeta}^n = (I_0 \text{ or } I_1) \cap F^{-1}(I_{\xi, \zeta}^{n-1}) \quad (11)$$

By induction, we assume that $I_{\xi, \zeta}^{n-1}$ is non-empty connected subinterval. Therefore, $F^{-1}(I_{\xi, \zeta}^{n-1})$ consists of two subintervals one in I_0 and the other in I_1 . Therefore, $(I_0 \text{ or } I_1) \cap F^{-1}(I_{\xi, \zeta}^{n-1})$ consists of a single interval. These intervals are nested. To prove this, we introduce slightly different notations.

$$\begin{aligned} I^n &= \{x \in I \mid F^{k-1}(x) \in I_0 \text{ for} \\ &\quad k \in \xi, F^{k-1}(x) \in I_1 \text{ for } k \in \zeta\} \\ I^{n-1} &= [x \in I \mid F^{k-1}(x) \in I_0 \text{ for} \\ &\quad k \in \xi_2, F^{k-1}(x) \in I_1 \text{ for } k \in \zeta_2] \end{aligned} \quad (12)$$

$\xi_2 = \xi$ if the integer n is not an entry in the set ξ , otherwise ξ_2 is obtained from the set ξ only by dropping the integer n . ζ_2 is defined similarly. Now,

$$\begin{aligned} \text{if } n \in \xi \text{ then } I^n &= I^{n-1} \cap \{x \in I \mid F^{n-1}(x) \in I_0\} \\ \text{if } n \in \zeta \text{ then } I^n &= I^{n-1} \cap \{x \in I \mid F^{n-1}(x) \in I_1\} \end{aligned} \quad (13)$$

Therefore, $I^n \subset I^{n-1}$. This implies that $\bigcap_n I^n$ is a nested sequences of subintervals and, therefore, it is non-empty. The subintervals I^n are contained in I_0 or I_1 and their lengths tend to zero as $n \rightarrow \infty$, and, therefore, the intersection is a unique point. Now, when $n \rightarrow \infty$, $I_{\xi, \zeta}^n$ is totally given by ξ , and ζ is just its' complement in the set of natural numbers. Therefore, to every subset $\xi \in \Sigma$, the set of all subsets natural numbers, there exists a unique point $x \in \Lambda$, the Cantor set. This implies that the itinerary, $S(x)$ is onto.

That distinct points of Cantor set, Λ correspond to different subsets of natural numbers can be seen from the fact that the subintervals $I^n(x)$, containing the point x , (constructed in

the previous section) tend to zero as 3^{-n+1} when $n \rightarrow \infty$. For $x \neq y$, consider neighbourhoods \aleph_x and \aleph_y of x and y such that $\aleph_x \cap \aleph_y = \emptyset$. Since the length of the subintervals $I^n \rightarrow 0$ as 3^{-n+1} , there exist an n_* , such that for $n > n_*$ $I^n(x) \in \aleph_x$ and $I^n(y) \in \aleph_y$.

In order to proof continuity, let us choose an ϵ . Now choose n such that $1/2^n < \epsilon$. Let $x \in \Lambda$ and $S(x) = (n_1, n_2, \dots, n_k, \dots)$. Consider all subintervals of the form $I_{\xi, \zeta}^n$ with n fixed but ξ and ζ varying. There are $2^n + 1$ such intervals. These subintervals are disjoint and Λ is contained in the union of them. Hence, we can choose a δ such that $|x - y| < \delta$, and $y \in \Lambda$, imply that $y \in I^n(x)$. Therefore, itinerary of the of the points, $S(x)$ and $S(y)$ agree on n entries. Now, from the metric on Σ it follows that

$$\|(S(x) - S(y))\| \leq 1/2^n < \epsilon \quad (14)$$

This proves continuity. Similarly, one can prove continuity of the inverse. Therefore, the itinerary $S(x)$ of points $x \in \Lambda$ is a homeomorphism between the Cantor set, Λ and the set of all subsets of natural number Σ . A set homeomorphic to a Cantor set is itself a Cantor set. This proves the main claim of the paper.

Now, we describe some interesting and important corrolaries of this result. Now, we prove that the "Reduction by Unity" map Ω on set Σ and Cantor map $F(x)$ on set Λ are topologically conjugate to each other. In other words,

$$S \circ F(x) = \Omega \circ S(x) \quad (15)$$

A point $x \in \Lambda$ is uniquely determined by intersection of nested sequence of intervals $\bigcap_n I^n$. These subintervals as argued earlier in the paper are determined by the itinerary $S(x)$.

$$\begin{aligned} & I_{\xi, \zeta}^n \\ &= (I_0 \quad \text{or} \quad I_1) \bigcap F^{-1}(I_{\xi_1, \zeta_1}^{n-1}) \end{aligned} \quad (16)$$

As explained earlier, ξ_1, ζ_1 are obtained from ξ and ζ by unit reduction map. Therefore,

$$\begin{aligned} F(x) &= F(I_{\xi, \zeta}^n, \quad \text{when} \quad n \rightarrow \infty) \\ &= I_{\xi_1, \zeta_1}^{n-1} \quad \text{when} \quad n \rightarrow \infty \end{aligned} \quad (17)$$

From here it follows that

$$\begin{aligned}
& S \circ F(I_{\xi, \zeta}^n, \quad \text{when } n \rightarrow \infty) \\
& = S(I_{\xi_1, \zeta_1}^{n-1}, \quad \text{when } n \rightarrow \infty) \\
& = (n_1 - 1, n_2 - 1, \dots, n_k - 1, \dots) \\
& = \Omega(n_1, n_2, \dots, n_k, \dots) \\
& = \Omega \circ S(x)
\end{aligned} \tag{18}$$

The "Reduction by Unity" map Ω on the set of all subsets of natural numbers, Σ is topologically conjugate to the Cantor map $F(x)$ on the Cantor set, Λ . This also proves that the evaporation process introduced earlier is also chaotic. This hypothetical evaporation process is a very generic example of chaos and, therefore, before we end the paper we mention the following interesting corollary. Let us construct a set of functions in the following way: to every configuration $\alpha = (n_1, n_2, \dots, n_k, \dots)$, we associate a function $h_\alpha(x)$

$$\begin{aligned}
h_\alpha(x) &= \exp(x) - \sum_{k \in \alpha} \frac{x^{k-1}}{(k-1)!} \\
&= \sum_{k \in \beta} \frac{x^{k-1}}{(k-1)!}
\end{aligned} \tag{19}$$

where $\alpha \cup \beta = \mathbf{N}$ and $\alpha \cap \beta = \emptyset$. When $\beta = \emptyset$, $\alpha = \mathbf{N}$, and therefore, according to the definition above $h_\emptyset(x) = 0$ and corresponds to set $\emptyset \in \Sigma$. It is clear from the definition that functions $h(x)$ are obtained from the exponential series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + \dots \tag{20}$$

by randomly dropping finite or infinite number of terms. We denote this set of functions as Ξ . We introduce on the set of functions, Ξ a metric in the following way,

$$\|(h_\alpha(x) - h_\beta(x))\| = \sum_{n=0}^{\infty} \frac{|h_\alpha^{(n)}(x) - h_\beta^{(n)}(x)|}{2^n} \Big|_{x=0} \tag{21}$$

here α and β are any subset in Σ and $h_\alpha^{(n)}(x)$ is the n^{th} derivative of $h_\alpha(x)$. It is clear that $\|(h_\alpha(x) - h_\beta(x))\| = 0$ imply $h_\alpha(x) = h_\beta(x)$ and the triangle inequality is satisfied. It is very

easily proven that the set of randomly diluted exponential functions, Ξ is a Cantor set in the above metric and the "Derivative" map on the functions of Ξ is chaotic.

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